COHOMOLOGY RINGS

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- Spheres:
$$H^*\big(S^n;\mathbb{Z}\big) \cong \mathbb{Z}[a_n] \,/\, \big(a_n^2 = 0\big)\,.$$
- Complex projective spaces:
$$H^*\big(\mathbb{C}P^n;\mathbb{Z}\big) \cong \mathbb{Z}[a_2] \,/\, \big(a_2^{n+1} = 0\big)\,.$$
- Real projective spaces:
$$H^*\big(\mathbb{R}P^n;\mathbb{Z}_2\big) \cong \mathbb{Z}_2[a_1] \,/\, \big(a_1^{n+1} = 0\big)\,.$$
- Unitary groups:
$$H^*\big(\mathrm{U}(n);\mathbb{Z}\big) \cong \Lambda_{\mathbb{Z}}[a_1, a_3 \ldots, a_{2n-1}].$$
- Special unitary groups:
$$H^*\big(\mathrm{SU}(n);\mathbb{Z}\big) \cong \Lambda_{\mathbb{Z}}[a_3, a_5 \ldots, a_{2n-1}].$$
- Special orthogonal groups:
$$H^*\big(\mathrm{SO}(2n+1);\mathbb{Q}\big) \cong \Lambda_{\mathbb{Q}}[a_3, a_7 \ldots, a_{4n-1}],$$
$$H^*\big(\mathrm{SO}(2n+2);\mathbb{Q}\big) \cong \Lambda_{\mathbb{Q}}[a_3, a_7, \ldots, a_{4n-1}, b_{2n+1}],$$

  (Note that $\mathrm{SO}(n)$ has 2-torsion for $n \geq 7$. See [1].)
$$H^*\big(\mathrm{SO}(n);\mathbb{Z}_2\big) \cong \bigotimes\nolimits_{i \text{ odd}} \mathbb{Z}_2[\beta_i]/\big(\beta_i^{p_i} = 0\big),$$
  where p_i is the smallest power of 2 such that $ip_i \geq n$. See [5, Theorem 3D.2].
- The exceptional Lie group G_2:
$$H^*\big(G_2;\mathbb{Q}\big) \cong \Lambda_{\mathbb{Q}}[h_3, h_{11}],$$
$$H^*\big(G_2;\mathbb{Z}\big) \cong \mathbb{Z}[h_3, h_{11}] \,/\, \big(h_3^4 = h_{11}^2 = h_3^2 h_{11} = 0\big)\,.$$
  Additively, we have
$$H^*\big(G_2;\mathbb{Z}\big) \cong \big(\mathbb{Z}, 0, 0, \mathbb{Z}h_3, 0, 0, \mathbb{Z}_2 h_3^2, 0, 0, \mathbb{Z}_2 h_3^3, 0, \mathbb{Z}h_{11}, 0, 0, \mathbb{Z}h_3 h_{11}\big)\,.$$
  See [1, Théorème 17.2].
- The exceptional Lie group F_4:
$$H^*\big(F_4;\mathbb{Q}\big) \cong \Lambda_{\mathbb{Q}}[h_3, h_{11}, h_{15}, h_{23}],$$
$$H^*\big(F_4;\mathbb{Z}\big) \cong H^*\big(G_2 \times S^{15};\mathbb{Z}\big) \otimes \mathbb{Z}[u_8, u_{23}] \,/\, \big(3u_8 = u_8^3 = u_{23}^2 = u_8 u_{23} = 0\big)\,.$$
  See [1, Théorème 19.2].

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- The exceptional Lie groups E_6, E_7, E_8:
$$H^*\big(E_6;\mathbb{Q}\big) \cong \Lambda_{\mathbb{Q}}[h_3, h_9, h_{11}, h_{15}, h_{17}, h_{23}],$$
$$H^*\big(E_7;\mathbb{Q}\big) \cong \Lambda_{\mathbb{Q}}[h_3, h_{11}, h_{15}, h_{19}, h_{23}, h_{27}, h_{35}],$$
$$H^*\big(E_8;\mathbb{Q}\big) \cong \Lambda_{\mathbb{Q}}[h_3, h_{15}, h_{23}, h_{27}, h_{35}, h_{39}, h_{47}, h_{59}].$$
See [4] and [3]. Furthermore, E_6, E_7, E_8 have no p torsion for $p \geq 7$, $p \geq 11$, and $p \geq 11$, respectively (see [2]).
- A projective K3 surface over $\mathbb{C}$:
$$H^*\big(K3;\mathbb{Z}\big) = \begin{cases} \mathbb{Z} & \text{in degrees } 0, 4, \\ \mathbb{Z}^{\oplus 22} & \text{in degree } 2, \\ 0 & \text{otherwise.} \end{cases}$$
The intersection pairing on $H^2(K3;\mathbb{Z})$ is even, unimodular, and has signature $(3, 19)$.

References

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