RESEARCH ABSTRACT

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1. Past and Current Research

The ultimate goal of my research is to provide an explicit construction for the Dirac operator on free loop spaces. In [11], Witten argued that the index of such a Dirac operator should yield the elliptic genus of the underlying space, and in [10], Taubes proved this using a localization argument, reducing the problem to constructing and computing the index of a local Dirac operator on the normal bundle to the constant loops. In my own research, I am working to construct a global version of the Dirac operator. My approach thus far has been to construct the Dirac operator for loop groups and for loops on homogeneous spaces. This allowed me to substitute an algebraic problem, now involving the representation theory of loop groups and infinite dimensional Lie algebras, for the intractable analysis on infinite dimensional spaces.

Working first with a compact Lie group, we can trivialize the tangent bundle by left translation. Recall that differentiation with respect to left invariant vector fields is equivalent to the infinitesimal right action on sections, and that the Levi-Civita connection for left invariant vector fields is half the Lie algebra bracket. The Levi-Civita Dirac operator on a Lie group is then of the form

$$\not\partial = \sum\nolimits_i c(X_i^*)\, r(X_i) + \tfrac{1}{2}\, c(X_i^*)\, \widetilde{\mathrm{ad}}(X_i), \tag{1}$$

where $\{X_i\}$ is a basis for the Lie algebra, $\{X_i^*\}$ is the dual basis, c and r are the Clifford and right actions respectively, and $\widetilde{\mathrm{ad}}$ is the lift of the adjoint representation from $\mathfrak{so}$ to $\mathfrak{spin}$. Since $\mathfrak{spin}$ consists of quadratic elements in the Clifford algebra, the term $\sum_i c(X_i^*)\, \widetilde{\mathrm{ad}}(X_i)$ is cubic, and in fact it is proportional to the fundamental 3-form given by $\omega(X, Y, Z) = \langle X, [Y, Z]\rangle$. All of the Dirac operators that I consider are variations of (1), using an infinite sum, a different constant in front of the cubic term, or a projection of the fundamental 3-form.

In [4], Gross, Kostant, Ramond, and Sternberg proved a generalization of the Weyl character formula in the case where $\mathfrak{g}$ is a semi-simple Lie algebra, and $\mathfrak{h}$ is a maximal rank reductive subalgebra. Letting V_λ and U_μ denote the irreducible $\mathfrak{g}$- and $\mathfrak{h}$-modules with highest weights λ and μ respectively, we have

$$V_\lambda \otimes \mathbb{S}^+_{\mathfrak{g}/\mathfrak{h}} - V_\lambda \otimes \mathbb{S}^-_{\mathfrak{g}/\mathfrak{h}} = \sum\nolimits_{c \in C} (-1)^c\, U_{c(\lambda+\rho_{\mathfrak{g}})-\rho_{\mathfrak{h}}}, \tag{2}$$

where $\rho_{\mathfrak{g}}$ and $\rho_{\mathfrak{h}}$ are half the sum of the positive roots of $\mathfrak{g}$ and $\mathfrak{h}$ respectively, $\mathbb{S}^\pm_{\mathfrak{g}/\mathfrak{h}}$ are the half-spin representations associated to the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$, and $C \subset W_{\mathfrak{g}}$ is a subset of the Weyl group of $\mathfrak{g}$ with one representative from each coset of $W_{\mathfrak{h}}$. The set of $\mathfrak{h}$-modules appearing on the right side of (2) is called an Euler number multiplet, as each multiplet contains $\chi(G/H)$ representations, where G and H are the corresponding Lie groups. Kostant further examined these

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multiplets in [5], constructing them explicitly as the kernel and cokernel of an algebraic Dirac operator $\not{\partial}_\lambda : V_\lambda \otimes \mathbb{S}^+_{\mathfrak{g}/\mathfrak{h}} \to V_\lambda \otimes \mathbb{S}^-_{\mathfrak{g}/\mathfrak{h}}$.

In [7], I turned this argument on its head, considering the geometric Dirac operator on the homogeneous space G/H. The index of this Dirac operator with values in the homogeneous vector bundle induced by an $\mathfrak{h}$-module U_μ is

$$\text{Index } \not{\partial}_{G/H} \otimes U_\mu = (-1)^c \, V_{c(\lambda+\rho_{\mathfrak{h}})-\rho_{\mathfrak{g}}} \, ,$$

if there exists $c \in W_{\mathfrak{g}}$ such that $c(\lambda + \rho_{\mathfrak{h}}) - \rho_{\mathfrak{g}}$ is a dominant weight for $\mathfrak{g}$, and the index vanishes if no such c exists. In other words, taking the index of the Dirac operator recovers the unique $\mathfrak{g}$-module V_λ such that U_μ lives in the multiplet corresponding to V_λ, and the index vanishes if U_μ does not live in any multiplet. Furthermore, I showed this representation V_λ is precisely the kernel or cokernel of the geometric version of Kostant's Dirac operator (which differs from the Levi-Civita Dirac operator by the constant in front of the cubic term).

In [8], I generalized the results of [4] and [5] to loop groups. In the Kac-Moody setting, I obtained a generalization of the Weyl-Kac character formula. Here, let $\mathcal{H}_{\lambda,h}$ and $\mathcal{U}_{\mu,k}$ denote the positive energy representations of $L\mathfrak{g}$ and $L\mathfrak{h}$ at levels h and k, with zero energy components of highest weights λ and μ respectively. The analogue of the spin representation $\mathbb{S}_{\mathfrak{g}/\mathfrak{h}}$ is the positive energy spin representation $\mathcal{S}_{L\mathfrak{g}/L\mathfrak{h}}$ of level $c_{\mathfrak{g}} - c_{\mathfrak{h}}$, where $c_{\mathfrak{g}}$ and $c_{\mathfrak{h}}$ are the values of the quadratic Casimir operators of $\mathfrak{g}$ and $\mathfrak{h}$ respectively acting in the adjoint representation. We then have

$$\mathcal{H}_{\lambda,h} \otimes \big(\mathcal{S}^+_{L\mathfrak{g}/L\mathfrak{h}} - \mathcal{S}^-_{L\mathfrak{g}/L\mathfrak{h}}\big) = \sum\nolimits_{w\in\mathcal{C}} (-1)^w \, \mathcal{U}_{w_{h+c_{\mathfrak{g}}}(\lambda+\rho_{\mathfrak{g}})-\rho_{\mathfrak{h}}\,,\, h+c_{\mathfrak{g}}-c_{\mathfrak{h}}} \, , \tag{3}$$

where $\mathcal{C} \subset \mathcal{W}_{\mathfrak{g}}$ is now a (usually infinite) subset of the affine Weyl group, and $w_{h+c_{\mathfrak{g}}}$ is the action of the affine Weyl group at the c-shifted level $h + c_{\mathfrak{g}}$. I then went on to construct the loop group version of Kostant's Dirac operator, $\not{\partial}_{\lambda,h} : \mathcal{H}_{\lambda,h} \otimes \mathcal{S}_{L\mathfrak{g}/L\mathfrak{h}} \to \mathcal{H}_{\lambda,h} \otimes \mathcal{S}_{L\mathfrak{g}/L\mathfrak{h}}$, and computing its square, I showed that the kernel of this operator is precisely the multiplet of representations appearing on the right side of (3).

In my Ph.D. thesis [6], I generalized my results in [7], constructing a geometric Dirac operator on the homogeneous loop space $L(G/H) = LG/LH$. In the infinite dimensional case, the primary obstacle is finding a suitable domain of spinors for the operator to act on. To this end, I used the Peter-Weyl space as a model for the space of L^2 functions on the central extension $\tilde{L}G$ of LG:

$$L^2(\tilde{L}G) := \bigoplus\nolimits_{\lambda,h} \mathcal{H}_{\lambda,h} \otimes \mathcal{H}^*_{\lambda,h},$$

where the sum is over all positive energy representations $\mathcal{H}_{\lambda,h}$ of $\tilde{L}G$, and $\mathcal{H}^*_{\lambda,h}$ is the reduced dual, a negative energy representation. To consider LH-invariant spinors, we must further tensor with some positive energy representation $\mathcal{U}_{\mu,k}$ in order that the total level vanish. Making sense of mixed tensor products of positive and negative energy representations, I showed that Bott's machinery (see [2]) for the index of homogeneous differential operators still holds in this context and proved that the index of a twisted Dirac operator on LG/LH is

$$\text{Index } \not{\partial}_{LG/LH} \otimes \mathcal{U}_{\mu,k} = (-1)^w \mathcal{H}_{w_{k+c_{\mathfrak{h}}}(\mu+\rho_h)-\rho_g\,,\, k+c_h-c_{\mathfrak{g}}} \, ,$$

if a suitable affine Weyl element w exists, or zero otherwise.

My thesis also considered the Dirac operator on the loop group LG, showing that as the level tends to infinity, these globals operators converge to Taubes' local

Dirac operator on the normal bundle to G in LG. Unfortunately, my argument did not work for the homogeneous loop space LG/LH since LH-invariance fixes the level at a single value. In subsequent unpublished research [9], I bypassed this difficulty by introducing *ghost fields*, using a variant of the BRST construction from string theory. If $\mathcal{H}$ is a LH-module, the BRST construction takes the relative Lie algebra cohomology $H^*(L\mathfrak{h}, h; \mathcal{H})$ as a model for the LH-invariant part of $\mathcal{H}$. At this point, the overall level of $\mathcal{H}$ must still vanish. However, if $\mathcal{U}$ and $\mathcal{V}$ are positive energy representations of LH, then we can compute the Lie algebra cohomology $H^*(L\mathfrak{h}, h; \mathcal{U} \otimes \mathcal{V}^*)$ using the so-called *split and flip* spectral sequence (see [3]) which has E_1 term

$$E_1^{p,q} = \big(H^p(L\mathfrak{h}^-; \mathcal{U}) \otimes H^q(L\mathfrak{h}^+; \mathcal{V}^*) \big)^{\mathfrak{h}}, \tag{4}$$

where $L\mathfrak{h}_{\mathbb{C}} = L\mathfrak{h}^- \oplus \mathfrak{h}_{\mathbb{C}} \oplus L\mathfrak{h}^+$ is a decomposition of $L\mathfrak{h}$ into its negative, zero, and positive Fourier components. This construction splits the Lie algebra cohomology into two factors of positive and negative energy respectively, and furthermore, this expression no longer requires that the total level vanish. So, adopting (4) as our model for the LH-invariant part of $\mathcal{U} \otimes \mathcal{V}^*$, we are now free to take the limit as the level tends to infinity, and we obtain

$$\lim_{h \to \infty} \operatorname{Index} \not{\partial}_{LG/LH} \otimes \mathcal{U} = \operatorname{Index} \not{\partial}_{G/H} \otimes \big(\operatorname{Sym}^*(L\mathfrak{g}^+/L\mathfrak{h}^+) \otimes \mathcal{U} \big),$$

giving the Witten genus for $\mathcal{U}$ trivial and other elliptic genera for suitable choices of $\mathcal{U}$. We can therefore approximate the elliptic genus to arbitrary precision as the index of a family of Dirac operators on homogeneous loop space.

2. Future Research Plans

Continuing my research on Dirac operators on loop spaces, the next step in the program is to consider the case of loops on projective varieties. In my research thus far, I have focused on homogeneous spaces G/H where G and H are of equal rank. This class includes the complex flag manifolds, and in particular complex projective space. Building on that, I intend to consider Dirac operators for loops on hyperplanes in complex projective space, taking advantage of the multiplicative properties of the index and elliptic genus. Such a construction may then generalize to other projective varieties, and in the long term, I hope to then make the leap from algebraic geometry to differential geometry and construct a global Dirac operator for loops on a smooth manifold.

On the algebraic side, I am interested in the non-commutative Weil algebra introduced in [1]. Given an ad-invariant inner product on a Lie algebra $\mathfrak{g}$, the tensor product $\mathcal{W}_{\mathfrak{g}} = U(\mathfrak{g}) \otimes \operatorname{Cl}(\mathfrak{g})$ of the universal enveloping algebra and the Clifford algebra admits a filtration, and the associated graded algebra is the usual Weil algebra $W_{\mathfrak{g}} = \operatorname{Sym}^*(\mathfrak{g}^*) \otimes \Lambda^*(\mathfrak{g}^*)$. Alekseev and Meinrenken showed that the exterior derivative d on $W_{\mathfrak{g}}$ can be expressed up to lower degree terms as an inner derivation on $\mathcal{W}_{\mathfrak{g}}$ by a cubic element $\not{\partial} \in \mathcal{W}_{\mathfrak{g}}$, where $\not{\partial}$ is a formal Dirac operator. In addition, they constructed a quantization map from $W_{\mathfrak{g}}$ to $\mathcal{W}_{\mathfrak{g}}$ intertwining the operators d and $\operatorname{ad} \not{\partial}$, thereby proving that the equivariant cohomology constructed using $\mathcal{W}_{\mathfrak{g}}$ agrees with the Weil model using $W_{\mathfrak{g}}$.

The Dirac operator on loop spaces that I have considered in my own research naturally lives in the looped version of the non-commutative Weil algebra. However, I have recently observed that $\operatorname{ad} \not{\partial}_{L\mathfrak{g}}$ does not give a quantization of the exterior derivative d on $W_{L\mathfrak{g}}$ but rather a quantization of the operator $d + u\,\iota_{\partial_\theta}$ acting on the product $W_{L\mathfrak{g}} \otimes \mathrm{Sym}^*(\mathbb{R}u)$, where u has degree 2. Restricting to the rotation invariant subspace, this setup looks suspiciously like the Cartan model for equivariant cohomology with respect to the S^1 action rotating the loops. I intend to investigate this further, examining how the quantization map behaves in the loop group case, as well as considering the semi-infinite version of the non-abelian Weil algebra.

I am also interested in studying the finite dimensional version of the *split and flip* construction I discussed above in (4). If $\mathfrak{g}$ can be decomposed as $\mathfrak{g}_{\mathbb{C}} = \mathfrak{n}^- \oplus \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{n}^+$, with $\mathfrak{n}^\pm$ nilpotent and conjugate to each other, then under appropriate conditions the relative Lie algebra cohomology splits as

$$H^*(\mathfrak{g}, \mathfrak{h};\, U \otimes V^*) \cong \bigl(\, H^*(\mathfrak{n}^-; U) \otimes H^*(\mathfrak{n}^+; V^*)\,\bigr)^{\mathfrak{h}}.$$

(This is analogous to the Hodge decomposition for the corresponding complex homogeneous space G/H.) I am curious whether this isomorphism can be made explicit using a relative version of Alekseev and Meinrenken's quantization map. If so, I hope to generalize that map to the loop group case to better justify (4).

Another problem stems from my generalization (3) of the Weyl-Kac character formula. In the loop group case, the multiplet of LH-modules corresponding to a given LG-module is now indexed by the quotient of the affine Weyl groups. If H is semi-simple, then this is a finite quotient, so we obtain a finite sum instead of the usual infinite sum offered by the Weyl-Kac character formula. I believe that this may imply an interesting relationship between the corresponding weight lattices, with potential consequences in number theory. Although my background is in geometry, not number theory, I would be interested in exploring this further, perhaps in collaboration with a colleague with a greater appreciation for the role of such lattices in number theory.

To consider spinors on loop spaces, the underlying manifold must possess not only a spin structure, but also a *string structure*, which requires that both w_2 and $\frac{1}{2}p_1$ vanish. If a manifold is not spin, the w_2 obstruction can often be canceled by twisting with a suitable complex line bundle to form a Spin^c structure. If the manifold does not admit a string structure, the required twisting is not by a line bundle, but rather by a gerbe, a higher order analogue of a vector bundle. Witten bypasses this difficulty by twisting the (right moving) spinors on the loop space by a second (left moving) spin representation obtained from a real, oriented vector bundle with the opposite obstructions w_2 and $\frac{1}{2}p_1$. I would like to study the finite dimensional version of this construction, considering the index of Dirac operators twisted by spin bundles associated to real vector bundles.

In addition to studying the structure and representations of loop groups and more general loop spaces, other areas of research that I am interested in pursuing in the coming years include vertex operator algebras, mirror symmetry, twisted K-theory, Seiberg-Witten theory, and Hamiltonian group actions.

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